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WAVELET ESTIMATION FOR DERIVATIVE OF A DENSITY IN A GARCH-TYPE MODEL

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Abstract: We consider the GARCH-type model $S = \sigma^2 Z$ where σ^2 and Z are independent random variables. We assume that the density f_{σ^2} of σ^2 is unknown with support $[0, 1]$ but differentiable where as the density f_S of S is bounded. We will also assume that the probability density function of the random variable Z is known and has the same distribution as the ν-fold product of independent random variables uniformly distibuted on the interval $[0, 1]$. We want to estimate the derivative of the density of σ^2 from n independent and identically distributed observations of S. We will construct adaptive and non-adaptive wavelet estimators for the derivative of the density and obtain sharp upper bounds on their mean integrated squared errors.

Keywords : Derivative of density estimation; GARCH-type model; Wavelets; Mean integrated squared error; Upper bound; Nonparametric inference.

Mathematics Subject Classification: 62G07, 62G20.

1 Introduction

We consider the GARCH-type model $S = \sigma^2 Z$ where σ^2 and Z are independent random variables. We assume that the density of σ^2 is unknown with support $[0, 1]$ but differentiable with a derivative square integrable on the interval $[0, 1]$, where as the density of Z is known. We want to estimate the derivative of the density of σ^2 from n independent and identically distributed (i.i.d.) observations of S. We will construct adaptive and non-adaptive wavelet estimators for the derivative of the density and study their properties.

Methods of nonparametric estimation of a density function and regression function are widely discussed in the literature (cf. Prakasa Rao (1983, 1999a)). It is known that the estimation of derivatives of a density are also of importance and interest to detect possible bumps and to detect monotonicity, concavity or convexity properties of the density function. Asymptotic properties of the kernel type estimators for the derivatives of density have been investigated earlier (cf. Prakasa Rao (1983)).

The Garch-type model is widely used in financial time series and the stochastic volatility σ^2 is unobserved (cf. Carrasco and Chen (2002)). Estimation of the probabilty density function of the stochastic volatility σ^2 is of extreme importance in analyzing financial data. The volatility function is nonnegative and can be assumed to be bounded in practice. Hence we can assume without loss of generality that the support of the stochastic volatility function is the interval $[0,1]$. Our aim in this paper is to discuss wavelet linear estimators for the derivative of a probability density function of the volatility function σ^2, assuming that it exists, from an i.i.d. sample of observations $\{S_i, 1 \leq i \leq n\}$. Estimators of density using wavelets was studied for independent and identically distributed random variables in Antoniadis et al. (1994), for some stationary dependent random variables in Leblanc (1996) and, for stationary associated sequences in Prakasa Rao (2003). Chaubey et al. (2006, 2008) extended these results to derivatives of density estimators for associated sequences and for negatively associated processes. The advantages and disadvantages of the use of wavelet based probability density estimators are discussed in Walter and Ghorai (1992) in the case of independent and identically distributed observations. However it was shown in Prakasa Rao (1996, 1999b) that one can obtain precise limits on the asymptotic mean squared error for a wavelet based linear estimator for the density function and its derivatives as well as some other functionals of the density. Tribouley (1995) studied estimation of multivariate densities using wavelet methods. Prakasa Rao (2000) investigated nonparametric estimation of the partial derivatives of a multivariate probability density. Donoho et al. (1996) investigated density estimation by wavelet thresholding. For a discussion on statistical modeling by wavelets, see Vidakovic (1999).

In recent papers, Chesneau and Doosti (2012) studied wavelet estimation of density for a GARCH model under various dependence structures and Chesneau (2013) investigated wavelet estimation of a density in a GARCH-type model leading to upper bounds on the mean integrated squared error. Shirazi et al. (2012) obtained wavelet based estimation of the derivative of a density by blockthresholding under random censorship. We will study estimation of the derivative of a density in GARCH-type model which can be considered as a generalization of multiplicative censoring model. Vardi (1989) (cf. Vardi and Zhang (1992)) introduced the multiplicative censoring model which unifies several models including nonparametric inference for renewal processes, non-parametric deconvolution problems and estimation of decreasing density functions. Chaubey et al. (2013) studied adaptive wavelet estimation of a density from mixtures under multiplicative censoring model generalizing the results in Prakasa Rao (2010). Asgharian et al. (2012) investigated asymptotic properties of the kernel density estimators under multiplicative censoring model. Andersen and Hansen (2001) studied density estimation for multiplicative censoring model using a series expansion approach. Chaubey et al. (2011) give a survey of recent results on linear wavelet density estimation and Chaubey et al. (2014) discuss adaptive wavelet estimation of a density from mixtures under multiplicative censoring.

2 Preliminaries on wavelets

A wavelet system is an infinite collection of translated and scaled versions of functions $\phi(.)$ and $\psi(.)$ called the *scaling function* and the *primary wavelet function* respectively. In the following discussion, we assume that $\phi(.)$ is real-valued. The function $\phi(x)$ is a solution of the equation

$$\phi(x) = \sum_{k=-\infty}^{\infty} C_k \phi(2x-k) \tag{2. 1}$$

with

$$\int_{-\infty}^{\infty} \phi(x)dx = 1 \tag{2. 2}$$

and the function $\psi(x)$ is defined by

$$\psi(x) = \sum_{k=-\infty}^{\infty} (-1)^k C_{-k+1} \phi(2x-k). \tag{2. 3}$$

The choice of the sequence $\{C_k\}$ determines the wavelet system. It is easy to see that

$$\sum_{k=-\infty}^{\infty} C_k = 2. \tag{2. 4}$$

Define

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), -\infty < j, k < \infty \tag{2. 5}$$

and

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), -\infty < j, k < \infty. \tag{2. 6}$$

Suppose the coefficients $\{C_k\}$ satisfy the condition

$$\begin{aligned} \sum_{k=-\infty}^{\infty} C_k C_{k+2\ell} &= 2 \text{ if } \ell = 0 \\ &= 0 \text{ if } \ell \neq 0. \end{aligned} \tag{2. 7}$$

It is known that, under some additional conditions on $\phi(.)$, the collection $\{\psi_{j,k}, -\infty < j, k < \infty\}$ is an orthonormal basis for $L^2(R)$, and $\{\phi_{j,k}, -\infty < k < \infty\}$ is an orthonormal system in $L^2(R)$, for each $-\infty < j < \infty$ (cf. Daubechies (1988, 1992)).

Definition 2.1: The scaling function ϕ is said to be r-regular for an integer $r \geq 1$, if for every nonnegative integer $\ell \leq r$, and for any integer $k \geq 1$,

$$|\phi^{(\ell)}(x)| \leq c_k(1+|x|)^{-k}, -\infty < x < \infty \tag{2. 8}$$

for some $c_k \geq 0$ depending only on k. Here $\phi^{(\ell)}(.)$ denotes the ℓ-th derivative of $\phi(.)$.

Definition 2.2: A *multiresolution analysis* of $L^2(R)$ consists of an increasing sequence of closed subspaces $\{V_j\}$ of $L^2(R)$ such that
(i) $\cap_{j=-\infty}^{\infty} V_j = \{0\}$;
(ii)$\bar{\cup}_{j=-\infty}^{\infty} V_j = L^2(R)$;
(iii) there is a scaling function $\phi \in V_0$ such that $\{\phi(x-k), -\infty < k < \infty\}$ is an orthonormal basis for V_0;
(iv) for all $h(.) \in L^2(R), -\infty < k < \infty, h(x) \in V_0 \Rightarrow h(x-k) \in V_0$; and
(v) $h(.) \in V_j \Rightarrow h(2x) \in V_{j+1}$.

Mallat (1989) has shown that, given any multiresolution analysis, it is possible to find a function $\psi(.)$ (called primary wavelet function) such that , for any fixed $j, -\infty < j < \infty$, the family $\{\psi_{j,k}, -\infty < k < \infty\}$ is an orthonormal basis of the orthogonal complement W_j of V_j in V_{j+1} so that $\{\psi_{j,k}, -\infty < j, k < \infty\}$ is an orthonormal basis of $L^2(R)$ (cf. Daubechies (1988, 1992)). When the scaling function $\phi(.)$ is r-regular, the corresponding multiresolution analysis is said to be r-regular.

Let $f \in L_2(R)$. The function f can be expanded in the form (cf. Daubechies (1992)):

$$\begin{aligned} f &= \sum_{k=-\infty}^{\infty} a_{s,k}\phi_{s,k} + \sum_{j=s}^{\infty}\sum_{k=-\infty}^{\infty} b_{j,k}\psi_{j,k} \\ &= P_s f + \sum_{j=s}^{\infty} D_j f \end{aligned} \tag{2. 9}$$

for any integer $-\infty < s < \infty$. Observe that the wavelet coefficients are given by

$$a_{s,k} = \int_{-\infty}^{\infty} f(x)\phi_{s,k}(x)dx \tag{2. 10}$$

and

$$b_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx. \tag{2. 11}$$

Suppose that the functions ϕ and ψ belong to C^r, the space of functions with r continuous derivatives for some $r \geq 1$, and have compact support contained in an interval $[-\delta, \delta]$ for some $\delta > 0$. It follows, from the Corollary 5.5.2 in Daubechies (1988), that the function $\psi(.)$ is orthogonal to polynomials of degree less than or equal to r. In particular

$$\int_{-\infty}^{\infty} \psi(x) x^{\ell} dx = 0, \ell = 0, 1, \ldots, r.$$

The above introduction to wavelets is based on Antoniades et al. (1994). For a detailed discussion, see Daubechies (1992). For a brief survey on wavelets, see Strang (1989).

3 Introduction to Sobolev spaces and Besov spaces

Let f be a function defined on the real line which is integrable on every bounded interval. It is said to be weakly differentiable if there exists a function g defined on the real line which is integrable on every bounded interval such that

$$\int_{x}^{y} g(u) du = f(y) - f(x).$$

The function g is defined almost everywhere and is called the *weak derivative* of f (cf. Hardle et al. (1998)). It is known that, if f is weakly differentiable with weak derivative g, then

$$\int_{-\infty}^{\infty} f(u) \phi'(u) du = -\int_{-\infty}^{\infty} g(u) \phi(u) du$$

for any $\phi \in D(R)$ where $D(R)$ denotes the space of infinitely differentiable functions, on the real line R, with compact support.

Definition 3.1: Let $1 \leq p \leq \infty$ and $m \geq 0$ be an integer. A function $f \in L_p(R)$ belongs to the *Sobolev space* $W_p^m(R)$, if it is m-times weakly differentiable and $f^{(m)} \in L_p(R)$. In particular $W_p^0(R) = L_p(R)$. The space $W_p^m(R)$ is equipped with the norm

$$||f||_{W_p^m} = ||f||_p + ||f^{(m)}||_p$$

where $||f||_p$ denotes the norm for $L_p(R)$.

Let $\tilde{W}_p^m(R) = W_p^m(R)$ if $1 \leq p < \infty$ and $\tilde{W}_\infty^m(R) = \{f : f \in W_\infty^m(R) : f^{(m)}$ uniformly continuous $\}$. Note that $\tilde{W}_p^0(R) = L_p(R), 1 \leq p < \infty$.

Let $f \in L_p(R)$ for some $1 \leq p \leq \infty$. Let $(\Delta_h f)(x) = f(x-h) - f(x)$ and define $\Delta_h^2 f = \Delta_h \Delta_h f$. For $t \geq 0$, define

$$\omega_p^1(f,t) = \sup_{|h|\leq t} ||\Delta_h f||_p$$

and

$$\omega_p^2(f,t) = \sup_{|h|\leq t} ||\Delta_h^2 f||_p.$$

Let $1 \leq q \leq \infty$. Suppose there exists a function $\epsilon(t)$ on $[0,\infty)$ such that $||\epsilon||_q^* < \infty$ where

(3. 1)
$$\begin{aligned} ||\epsilon||_q^* &= (\int_0^\infty t^{-1}|\epsilon(t)|^q dt)^{1/q}, \text{ if } 1 \leq q < \infty \\ &= ess\sup_t |\epsilon(t)|, \text{ if } q = \infty. \end{aligned}$$

Definition 3.2: Let $1 \leq p, q \leq \infty$ and $s = n+\alpha$ where $n \geq 0$ is an integer and $0 < \alpha \leq 1$. The *Besov space* $B_{p,q}^s$ is the space of all functions f such that $f \in W_p^n(R)$ and $\omega_p^2(f^{(n)},t) = \epsilon(t)t^\alpha$ where $||\epsilon||_q^* < \infty$.

For properties of Besov spaces, see Meyer (1990) and Triebel (1992) (cf. Leblanc (1996), Hardle et al. (1998)).

Suppose that the function f belongs to the Besov class

$$F_{s,p,q}(L) = \{f \in B_{p,q}^s, ||f||_{B_{p,q}^s} \leq L\}$$

for some $0 < s < r+1, p \geq 1$ and $q \geq 1$, where

$$||f||_{B_{p,q}^s} = ||P_0 f||_p + [\sum_{j\geq 0}(||D_j f||_p 2^{js})^q]^{1/q}.$$

Given a double indexed sequence $\{\gamma_{j,k}\}$ define the norm

(3. 2)
$$||\gamma_{j,.}||_{\ell_p} = (\sum_k \gamma_{j,k}^p)^{1/p}.$$

In view of the representation (2.9), it be can shown that the function $f \in B_{p,q}^s$ if and only if

(3. 3)
$$||a_{s,.}||_{\ell_p} < \infty, \text{ and } (\sum_{j\geq s}[||b_{s,.}||_{\ell_p} 2^{j(s+(1/2)-(1/p))}]^q)^{1/q} < \infty.$$

Let $\phi(.)$ be a scaling function as defined earlier. Define

$$\theta_\phi(x) = \sum_{k=-\infty}^{\infty} |\phi(x-k)|.$$

Suppose the following conditions hold:

(C1) The $ess\sup_x \theta_\phi(x) < \infty$ where

$$ess\sup_x g(x) = \inf\{y : \lambda([x : g(x) > y]) = 0\}$$

and λ is the Lebesgue measure on the real line.
(C2) There exists a bounded nondecreasing function $\Phi(.)$ such that $|\phi(u)| \leq \Phi(|u|)$ almost every where and

$$\int_0^\infty |u|^r \Phi(|u|)du < \infty.$$

for some integer $r \geq 0$.

Lemma 3.1: Suppose that the scale function $\phi(.)$ is such that the collection $\{\phi(x-k), -\infty < k < \infty\}$ is an orthonormal system in $L_2(R)$ and the spaces $V_j, -\infty < j < \infty$ are nested. Further suppose that the function ϕ satisfies the condition (C2) and it is $r+1$ times weakly differentiable. If $\phi^{(r+1)}$ satisfies the condition (C1), then the norm $||.||_{B^s_{p,q}}$ is equivalent to the norm $||.||'_{B^s_{p,q}}$ in the space of the wavelet coefficients for all s, p, q such that $0 < s < r+1$ and $1 \leq p, q \leq \infty$ where

$$||f||'_{B^s_{p,q}} = ||a_0||_p + (\sum_{j=0}^{\infty}(2^{j(s+(1/2)-(1/p))}||b_j||_p)^q)^{1/q}.$$

Here $||a_0||_p$ denotes $[\sum_{k=-\infty}^{\infty} |a_{0,k}|^p]^{1/p}$ and $||b_j||_p$ denotes $[\sum_{k=-\infty}^{\infty} |b_{j,k}|^p]^{1/p}$.

For a proof of Lemma 3.1, see Theorem 9.6 in Hardle et al. (1998), p.123.

4 Estimation of the d-th derivative of a probability density function

Let $\{Y_i, 1 \leq i \leq n\}$ be independent and identically distributed random variables with probability density function f which is d-times differentiable. Suppose that the derivative $f^{(d)}$ is bounded and has compact support. Suppose that $f^{(d)} \in L_2(R)$. Let us first consider the estimation of the probability density function f. A wavelet based density estimator of the density function f can be motivated in the following way from the expansion given in (2.9) (cf. Prakasa Rao (2003)). We can estimate $f(x)$ by $\hat{f}(x)$ where

$$\hat{f}(x) = \sum_{k \in N_s} \alpha_{s,k}\phi_{s,k}(x) \quad (4.\ 1)$$

where

$$\alpha_{s,k} = \frac{1}{n}\sum_{i=1}^{n}\phi_{s,k}(Y_i). \tag{4. 2}$$

Here N_s is the set of integers k such that $supp(f) \cap supp(\phi_{s,k})$ is nonempty. Since the functions f and ϕ have compact supports, the cardinality of the set N_s is finite and it is of the order $O(2^s)$.

Let us now consider the problem of estimation of the derivative $f^{(d)}$ of f. As in Prakasa Rao (1996), we assume that the scaling function $\phi(.)$ generates a r-regular multiresolution analysis for some $r \geq 2$ and that there exists $C_m \geq 0$ and $\beta_m \geq 0$ such that

$$|f^{(m)}(x)| \leq C_m(1+|x|)^{-\beta_m}, \quad 0 \leq m \leq r. \tag{4. 3}$$

This assumption implies that that the derivative $\phi^{(d)}$ is bounded for every $d \geq 0$(cf. Prakasa Rao (1996)). Furthermore the projection of $f^{(d)}$ on V_s is

$$f_s^{(d)}(x) = \sum_{k\in N_s} a_{s,k}\phi_{s,k}(x) \tag{4. 4}$$

where

$$a_{s,k} = (-1)^d \int_{-\infty}^{\infty} f(x)\phi_{s,k}^{(d)}(x)dx.$$

The equation given above can be justified by using integration by parts since the function $\phi(.)$ is r-regular (cf. Prakasa Rao (1996)). This expression motivates the following estimator for $f^{(d)}(x)$:

$$\hat{f}_s^{(d)}(x) = \sum_{k\in N_s} \hat{a}_{s,k}\phi_{s,k}(x) \tag{4. 5}$$

where

$$\hat{a}_{s,k} = \frac{(-1)^d}{n}\sum_{i=1}^{n}\phi_{s,k}^{(d)}(Y_i).$$

Note that the estimator defined above reduces to the density estimator given in (4.1) for $d = 0$. We now rewrite the expression for the estimator $\hat{f}_s^{(d)}(x)$ in a slightly different form. Note that

$$\begin{aligned} \hat{f}_s^{(d)}(x) &= \sum_{k\in N_s} \hat{a}_{s,k}\phi_{s,k}(x) \\ &= \sum_{k\in N_s} [\frac{(-1)^d}{n}\sum_{i=1}^{n}\phi_{s,k}^{(d)}(Y_i)]\phi_{s,k}(x) \\ &= \frac{(-1)^d}{n}\sum_{i=1}^{n}\sum_{k\in N_s}\phi_{s,k}^{(d)}(Y_i)\phi_{s,k}(x) \end{aligned} \tag{4. 6}$$

$$
\begin{aligned}
&= \frac{(-1)^d}{n}\sum_{i=1}^{n}\sum_{k\in N_s} 2^{(s/2)+ds}\phi^{(d)}(2^sY_i-k)2^{s/2}\phi(2^sx-k)\\
&= \frac{(-1)^d}{n}\sum_{i=1}^{n}[\sum_{k\in N_s}\phi^{(d)}(2^sY_i-k)\phi(2^sx-k)]2^{s+ds}\\
&= \frac{(-1)^d}{n}\sum_{i=1}^{n}K^{(d)}(2^sY_i,2^sx)2^{s+ds}\\
&= \frac{(-1)^d}{n}\sum_{i=1}^{n}K_s^{(d)}(Y_i,x)
\end{aligned}
$$

where

$$K_s(x,y) = 2^s\ K(2^sx,2^sy)$$

and

$$K(x,y) = \sum_{k\in N_s}\phi(x-k)\phi(y-k).$$

Here $K_s^{(d)}(x,y)$ denotes the d-th partial derivative of $K_s(x,y)$ with respect to x.

5 Estimation of the wavelet coefficients

Consider the Garch-type model $S=\sigma^2Z$ as described earlier. Suppose the random variable Z has the probability density function

$$f_Z(z) = \frac{1}{(\nu-1)!}(-\log z)^{\nu-1}, 0\le z\le 1$$

where ν is a known positive integer. It is easy to see that, if $\nu=1$, then the density f_z is the standard uniform density function and in general the F_Z is the density of $\Pi_{i=1}^{\nu}U_i$ where $U_1,\ldots,U_\nu$ are i.i.d. random variables with standard uniform distribution.

We will now consider the problem of estimation of the first derivative of the density f_{σ^2} hereafter for simplicity based on i.i.d. observations on the random variable S. Similar results can be obtained for the estimation of the d-th derivative of the density f_{σ^2} for $d\ge 1$. Let N be an integer such that $N>10-\nu$ and ϕ and ψ be the scaling function and the primary wavelet function. Suppose the support of ϕ and the support of the function ψ are contained in the interval $[1-N,N]$ and the functions ϕ and ψ are in class $C^{\nu+1}$. Let

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^jx-k)$$

and

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Under these conditions, there exists an integer τ such that $2^\tau > 2N$ and the a family of functions $\{\phi_{j,k}, k = 0, \ldots, 2^j - 1; \psi_{j,k}, j \in N_0 - 0, \ldots, \tau - 1, k = 0, \ldots, 2^j - 1\}$ is an orthonormal basis of $L^2([0,1])$ (cf. Cohen et al. (1993)). For any integer $\ell \geq \tau$ and $h \in L^2([0,1])$, we can expand the function h as

$$h(x) = \sum_{k=0}^{2^\ell - 1} \alpha_{\ell,k}\phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty}\sum_{k=0}^{2^\ell - 1} \beta_{j,k}\psi_{j,k}(x)$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h given by

$$\alpha_{j,k} = \int_0^1 h(x)\phi_{j,k}(x)dx$$

and

$$\beta_{j,k} = \int_0^1 h(x)\psi_{j,k}(x)dx.$$

We will first state few lemmas which will be used in the sequel.

Lemma 5.1: For any positive integer k and any function $h \in C^k[0,1]$, let

$$G(h)(x) = -xh'(x)$$

and

$$G_k(h)(x) = G(G_{k-1}(h))(x).$$

Define

$$T(h)(x) = (xh(x))';\ T_k(h)(x) = T(T_{k-1}(h))(x), 0 \leq x \leq 1.$$

Then

$$(i) f_{\sigma^2}(x) = G_\nu(f_S)(x), 0 \leq x \leq 1;$$

and

(ii)for any $h \in C[0,1]$,

$$\int_0^1 f_\sigma^2(x)h(x)dx = \int_0^1 f_S(x)T_\nu(h)(x)dx.$$

For a proof of this lemma, see Chesneau (2013).

Since the derivative f'_{σ^2} is assumed to be square integrable on $[0,1]$, we can expand using the wavelet basis $\{\phi_{jk}\}$. Let

$$a_{j,k} = \int_0^1 f'_{\sigma^2}(x)\phi_{jk}(x)dx, j \geq \tau, k = 0, \ldots, 2^j - 1.$$

Following the assumptions made on the wavelet basis and the density function f_{σ^2}, it follows that

$$\begin{aligned} a_{j,k} &= \int_0^1 f'_{\sigma^2}(x)\phi_{j,k}(x)dx \\ &= -\int_0^1 f_{\sigma^2}(x)\phi'_{j,k}(x)dx \\ &= -\int_0^1 f_S(x)T_\nu(\phi'_{j,k})(x)dx \ \text{ (by Lemma 5.1)} \\ &= -E[T_\nu(\phi'_{j,k})(S)]). \end{aligned}$$

In view of the above relation, we can estimate the coefficient $a_{j,k}$ by

$$\hat{a}_{j,k} = \frac{1}{n}\sum_{i=1}^{n} T_\nu(\phi'_{j,k})(S_i) \tag{5. 1}$$

and we can estimate the the coefficient $b_{j,k}$ by

$$\hat{b}_{j,k} = \frac{1}{n}\sum_{i=1}^{n} T_\nu(\psi'_{j,k})(S_i). \tag{5. 2}$$

This method of estimation follows the ideas developed in Abbaszadeh et al. (2012).

Let $M > 0, s > 0, p \geq 1$ and $r \geq 1$. Let $B^s_{p,r}(M)$ be the class of functions h such that there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients satisfy the condition

$$2^{\tau(\frac{1}{2}-\frac{1}{p})}(\sum_{k=0}^{2^\tau-1} |\alpha_{\tau,k}|^p)^{1/p} + [\sum_{j=\tau}^{\infty}(2^{j(s+\frac{1}{2}-\frac{1}{p})}(\sum_{k=0}^{2^j-1} |\beta_{j,k}|^p)^{\frac{1}{p}})^r)]^{\frac{1}{r}} \leq M^*.$$

Such a class of functions is called the *Besov Ball* with smoothness parameter$s$ and norm parameters p and r.

Suppose that $f'_{\sigma^2} \in B^s_{p,r}(M)$ with $p \geq 2$. Define the *linear estimator* of f'_{σ^2} by

$$\hat{f}'(x) = \sum_{k=0}^{2^{j_0}-1} \hat{a}_{j_0,k}\phi_{j_0,k}(x), x \in [0,1] \tag{5. 3}$$

where $\hat{a}_{j,k}$ is as defined in (5.1) and j_0 is an integer which will be chosen later. Discussion on such linear estimators for density function is given in Prakasa Rao (1983,1996) and a survey on such estimators for various density models is given in Chaubey et al. (2011). Let us also define the *thresholding estimator* of f'_{σ^2} by

$$(5.\ 4)\qquad \tilde{f}'(x) = \sum_{k=0}^{2^\tau -1} \hat{a}_{\tau,k}\phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1} \hat{b}_{j,k} I[|\hat{b}_{j,k}| \geq \kappa\lambda_j]\psi_{j,k}(x), x \in [0,1]$$

where $\hat{a}_{\tau,k}$ is as defined by (5.1), $\hat{b}_{j,k}$ is as defined by (5.2), j_1 is the integer satisfying

$$\frac{1}{2}(\frac{n}{\log n})^{1/(2(\nu+1)+1)} < 2^{j_1} \leq (\frac{n}{\log n})^{1/(2(\nu+1)+1)},$$

$$\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{\log n}{n}}$$

and κ is a constant to be chosen. Here $I[A]$ denotes the indicator function for the set A. The thresholding estimator takes into account the contribution of large unknown wavelet coefficients of f'_{σ^2} in the wavelet expansion of f'_{σ^2}. Construction of such estimators for the estimation of a probability density were first discussed in Donoho et al. (1996). Chaubey et al. (2013) study similar estimators for density under multiplicative censoring.

We now obtain upper bounds on the integrated mean square error for the estimators $\hat{f}'$ and $\tilde{f}'$ of f'_{σ^2}.

Theorem 5.1: Suppose that $f'_{\sigma^2} \in B^s_{p,r}(M)$ for some $s > 0, p \geq 2$ and $r \geq 1$. Let $\hat{f}'$ be as defined by (5.3)and j_0 be an integer such that

$$\frac{1}{2}n^{1/(2s+2(\nu+1)+1)} < 2^{j_0} \leq n^{1/(2s+2(\nu+1)+1)}.$$

Then there exists a constant $C > 0$ such that

$$E[\int_0^1 (\hat{f}'(x) - f'_{\sigma^2}(x))^2 dx] \leq C n^{-2s/(2s+2(\nu+1)+1)}.$$

Theorem 5.2: Suppose that $f'_{\sigma^2} \in B^s_{p,r}(M)$ for some $s > 0, p \geq 2$ and $r \geq 1$ or $1 \leq p < 2$ and $s > \frac{2(\nu+1)+1}{p}$.Let $\tilde{f}'$ be as defined by (5.4). Then there exists a constant $C > 0$ such that

$$E[\int_0^1 (\tilde{f}'(x) - f'_{\sigma^2}(x))^2 dx] \leq C(\frac{\log n}{n})^{2s/(2s+2(\nu+1)+1)}.$$

6 Proofs of Theorem 5.1 and Theorem 5.2

Since the proofs of Theorems 5.1 and 5.2 are similar to those in Cheneau (2013) and are simpler due to the i.i.d. nature of the observations on the random variable S, we only sketch them. We will first prove another lemma.

Lemma 6.1: For any integer $j \geq \tau$ and $k \in \{0, \ldots, 2^j - 1\}$, let

$$a_{j,k} = \int_0^1 \phi_{j,k}(x) f'_{\sigma^2}(x) dx$$

and define the operator T as given earlier, that is, for any function $h \in C^\ell[0,1]$,

$$T(h)(x) = (xh(x))'.$$

Then there exist a constant $C > 0$ depending on ν and the wavelet basis such that

$$(i) E(T_\nu(\phi'_{j,k})(S_1)) = -a_{j,k};$$

$$(ii) E[(T_\nu(\phi'_{j,k})(S_1))^2] \leq C 2^{2(\nu+1)j};$$

and

$$Var[\sum_{i=1}^n T_\nu(\phi'_{j,k})(S_1) \leq nC 2^{2(\nu+1)j}.$$

Similar results hold for the function ψ and the corresponding wavelet coefficients $b_{j,k}$.

Proof : Note that, for any $u \in \{0, \ldots, \nu\}$, the function $\phi^{(u)}_{j,k}(x)$, the u-th derivative of the function $\phi_{j,k}$ at x, is given by

$$\phi^{(u)}_{j,k}(x) = 2^{(2u+1)j/2} \phi^{(u)}(2^j x - k).$$

Following the computations in Proposition 6.1 of Cheneau (2013), it is easy to check that

$$\sup_{0 \leq x \leq 1} |T_\nu(\phi'_{j,k})(x)| \leq C 2^{(2(\nu+1)+1)j/2}$$

for some constant $C > 0$ depending on ν and the wavelet basis. Since the support of the random variable S_1 is bounded, it can be checked that there exists a constant $C > 0$ such that

$$E[(T_\nu(\phi'_{j,k})(S_1))^2] \leq C 2^{2(\nu+1)j}$$

and

$$Var[T_\nu(\phi'_{j,k})(S_1)] \leq C 2^{2(\nu+1)j}.$$

Hence

$$Var(\sum_{i=1}^{n} T_\nu(\phi'_{j,k})(S_i)) \leq Cn2^{2(\nu+1)j}.$$

As a consequence of Lemma 6.1, the following upper bounds can be obtained for the estimators $\hat{a}_{j,k}$ of wavelet coefficients $a_{j,k}$. There exists a constant $C > 0$ depending on ν and the wavelet basis such that, for any $j \geq \tau$ and $k \in \{0, \ldots, 2^j - 1\}$,

$$E(\hat{a}_{j,k} - a_{j,k})^2 \leq Cn^{-1}2^{2(\nu+1)j} \tag{6. 1}$$

and

$$E(\hat{a}_{j,k} - a_{j,k})^4 \leq Cn^{-1}2^{(4(\nu+1)+1)j}. \tag{6. 2}$$

For any $j \in \{\tau, \ldots, j_1\}$ and $k \in \{0, \ldots, 2^j - 1\}$, consider the wavelet coefficients $b_{j,k}$ and the corresponding estimators $\hat{b}_{j,k}$ as defined earlier. Define $\lambda_j = 2^{(\nu+1)j}(\frac{\log n}{n})^{1/2}$. Let

$$U_i = T_\nu(\psi'_{j,k})(S_i) - b_{j,k}, i = 1, \ldots, n.$$

The sequence $U_i, 1 \leq i \leq n$ is a sequence of mean zero independent random variables such that

$$|U_i| \leq C2^{(2(\nu+1)+1)j/2}, 1 \leq i \leq n$$

where C is a constant depending on the wavelet basis. Applying Bernstein's inequality or Hoeffding's inequality for bounded independent random variables (cf. Lin and Bai (2010), p.74), it follows that

$$\begin{aligned} P(|\hat{b}_{j,k} - b_{j,k}| \geq \kappa\lambda_j/2) &= P(\sum_{i=1}^{n} U_i \geq n\kappa\lambda_j/2) \\ &\leq 2\exp\{-n^2\kappa^2\lambda_j^2/8nC^2 2^{(2(\nu+1)+1)j}\} \\ &= 2\exp\{-n\log n\, \kappa^2/(8C^2\, 2^j)\}. \end{aligned}$$

Proof of Theorem 5.1: Observe that, for any $x \in [0, 1]$,

$$\hat{f}'(x) - f'_{\sigma^2}(x) = \sum_{k=0}^{2^{j_0}-1} (\hat{a}_{j,k} - a_{j,k})\phi_{j,k}(x) - \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} b_{j,k}\psi_{j,k}$$

and hence

$$E[\int_0^1 (\hat{f}'(x) - f'_{\sigma^2}(x))^2 dx] = \sum_{k=0}^{2^{j_0}-1} E[(\hat{a}_{j,k} - a_{j,k})^2] + \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} b_{j,k}^2$$

by the orthonormality of the wavelet basis. In view of inequality (6.1), we get that

$$\sum_{k=0}^{2^{j_0-1)}} E[(\hat{a}_{j,k} - a_{j,k})^2] \leq C2^{j_0(2(\nu+1)+1)}n^{-1} \leq Cn^{-2s/(2s+(2(\nu+1)+1)}.$$

Since $p \geq 2$, it follows that $B^s_{p,r}(M) \subset B^s_{2,\infty}(M)$, and hence

$$\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1} b^2_{j,k} \leq C2^{-2j_0 s} \leq Cn^{-2s/(2s+(2(\nu+1)+1)}.$$

Therefore

$$E[\int_0^1 (\hat{f}'(x) - f'_{\sigma^2}(x))^2 dx] \leq Cn^{-2s/(2s+(2(\nu+1)+1)}.$$

Proof of Theorem 5.2: Following the notation used earlier, we can expand the function $f'_{\sigma^2}(x)$, using the wavelet basis defined earlier, as

$$f'_{\sigma^2}(x) = \sum_{k=0}^{2^\tau-1} a_{j,k}\phi_{\tau,k}(x) + \sum_{j=\tau}^{\infty}\sum_{k=0}^{2^j-1} b_{j,k}\psi_{j,k}(x), 0 \leq x \leq 1$$

and hence

$$\tilde{f}'(x)-f'_{\sigma^2}(x) = \sum_{k=0}^{2^\tau-1}(\hat{a}_{j,k}-a_{j,k})\phi_{\tau,k}(x)+\sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}(\hat{b}_{j,k}I[|\hat{b}_{j,k}| > \kappa\lambda_j]-b_{j,k})\psi_{j,k}(x)-\sum_{j=j_1+1}^{\infty}\sum_{k=0}^{2^j-1} b_{j,k}\psi_{j,k}(x).$$

Since the wavelet basis is an orthonormal basis of $L^2[0,1]$, it follows that

$$E[\int_0^1 (\tilde{f}'(x) - f'_{\sigma^2}(x))^2 dx] = R_1 + R_2 + R_3$$

where

$$R_1 = \sum_{k=0}^{2^\tau-1} E[(\hat{a}_{j,k} - a_{j,k})^2],$$

$$R_2 = \sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1} E[(\hat{b}_{j,k}I[|\hat{b}_{j,k}| > \kappa\lambda_j] - b_{j,k})^2],$$

and

$$R_3 = \sum_{j=j_1+1}^{\infty}\sum_{k=0}^{2^j-1} b^2_{j,k}.$$

Following arguments given in Chesneau (2013), which are much easier here due to the i.i.d. structure for the random sequence $\{S_i, i \geq 1\}$, it can be shown that

$$R_1 \leq C(\frac{\log n}{n})^{2s/(2s+2(\nu+1)+1)}$$

$$R_2 \leq C(\frac{\log n}{n})^{2s/(2s+2(\nu+1)+1)}$$

and

$$R_3 \leq C(\frac{\log n}{n})^{2s/(2s+2(\nu+1)+1)}.$$

Hence

$$E[\int_0^1 (\tilde{f}'(x) - f'_{\sigma^2}(x))^2 dx] \leq C(\frac{\log n}{n})^{2s/(2s+2(\nu+1)+1)}.$$

Remarks : We have obtained upper bounds for the integrated mean square error for the linear and threshold estimators of the derivative of the density of f_{σ^2}. The results can be easily extended to the d-th derivative of the density by analogous arguments for $d \geq 1$ and the bounds are obtained by replacing the term $\nu+1$ by $\nu+d$ throughout the calculations. We can also get bounds analogous to those given above for the integrated mean squared error for the adaptive as well as linear estimators for the derivative of the density in case the process $\{S_i, i \geq 1\}$ is exponentially strong mixing as in Chestneau (2013). The bounds obtained in Chestneau (2013) will continue to hold for the d-th derivative of f_{σ^2} by replacing ν by $\nu + d$ in his results. We are omitting the details as the arguments are similar for proving these results. As has been pointed out by Chesneau (2013), upper bound for the integrated mean square of the threshold estimator is sharp in the sense that it is close to the corresponding bound for the linear wavelet estimator.

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