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Discrete Linear Sylvester Repetitive Process

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Abstract. Repetitive Processes are a special class of dynamical systems which involve propagation of information in two independent directions that are having theoretical importance and practical relevance. Discrete Sylvester systems have applications in control theory, optimal filters, system theory, differential games, power systems and signal processing. So considering repetitive processes described by discrete Sylvester systems is an interesting problem because of applicability. These systems can not be controlled by conventional system theoretic approach. Long wall cutting problems, metal rolling operations, Iterative learning control (ILC) schemes and iterative algorithms for solving non linear optimal control problems are some of the important applications of Repetitive Processes. This paper characterizes controllability properties to design control schemes based on linear matrix inequality (LMI) for Discrete Linear Sylvester Repetitive Process.

1 Introduction

Linear repetitive processes are a special class of dynamical systems which are of theoretical relevance and applications importance. The essential unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length.

On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. The essential unique control problem for these processes arises directly from the explicit interaction between successive pass profiles. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction. Such behaviour is easily generated in simulation studies and in experiments on scaled models longwall coal cutting and metal rolling operations problems. In recent past applications of these processes extended to iterative learning control (ILC) schemes [5] and iterative algorithms for solving nonlinear dynamic optimal control problems which became essential tools for stability analysis and convergence analysis. Physical examples of repetitive processes include longwall coal cutting and metal rolling operations

The discrete matrix Sylvester systems are generalizations of the discrete Lyapunov systems that arise in the construction of Lyapunov functions, parametric optimization problems and linear complementarity problems which have applications in signal process, population dynamics, time series

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analysis and computational mechanics. In 1997 Murthy, KN and Anand, P V S [21] established the necessary and sufficient conditions for the controllability and observability of the time varying system associated with the continuous matrix Liapunov system. In 2005 Murthy and Anand [22] established necessary and sufficient conditions for the controllability and observability of Liapunov type matrix difference system.

The history of linear matrix inequalities(LMIs) in the analysis of dynamical systems goes back to the famous " The General Problem of Stability of Motion " of Lyapunov, 1892. In 1940's. Lur'e, Postnikov addressed the problem of stability of a control system with a nonlinearity in the form of LMIs. In 1940's Small LMIs solved by hand were used in application of Lyapunov's methods to real control engineering problems. In 1960's, Yakubovich, Popov, Kalman, and many other researchers succeeded in reducing the solution of the LMIs that arose in the problem of Lur'e to simple graphical criteria, using positive-real (PR) lemma and these LMI's can be solved by solving algebraic Riccati equations. Due to the advent of computers LMI's are solved by using convex programming and interior point algorithms. J.B. Edwards [8] studied Stability problems in the control of multi pass processes, in 1974. In 1992 E. Rogers and D. H. Owens [28] have done stability analysis for linear repetitive processes and again in 2004, B. Sulikowski K. Galkowski E. Rogers D. H. Owens studied output feedback control of discrete linear repetitive processes using LMI setting. Subsequently in the last decade many authors started working on discrete and continuous repetitive processes using LMIs. One unique feature of repetitive processes is, it is possible define physically meaningful control laws for them. For example, in the iterative learning control application, one such family of control laws is composed of, state or output based, feedback control action on the current pass combined with information "fed forward" from the previous pass, or trial in the iterative learning control context has already been generated and is therefore available for use. It is always desired "to design control laws for stability so that performance can be guaranteed". In the case of repetitive processes, it is physically meaningful to define the current pass output error as the difference, at each point along the pass, between a specified reference trajectory for that pass, which in most cases will be the same on each pass, and the actual pass-profile produced. Then we can define a current pass error actuated controller which uses the generated error vector to construct the current pass control input vector. In 2000 S. E. Benton [3] shown that "except in a few very restrictive special cases, the controller used must be actuated by a combination of current pass information and feed forward information from the previous pass to guarantee even stability along the pass closed loop". In the iterative learning control (ILC) applications, the previous pass (or trial in the ILC setting) output vector is an obvious signal to use as "feed forward" action. Now consider Discrete Linear Sylvester Repetative Process

$$\Delta T_{l+1}(n) = AT_{l+1}(n) + T_{l+1}(n)B + AT_{l+1}(n)B + KU_{l+1}(n) + K_0 Z_l(n) \tag{1.1}$$

$$Z_{l+1}(n) = CT_{l+1}(n) + DU_{l+1}(n) + D_0 Z_l(n), 0 \leq n \leq \infty, l \geq 0 \tag{1.2}$$

Here on pass l, $T_l(n)$ is the $s \times s$ state matrix, $Z_l(n)$ is the $m \times s$ matrix pass profile,and $U_l(n)$) is the $s \times m$ matrix of control inputs. Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0 (< 0)$ denotes a real symmetric positive (negative) definite matrix.

This paper is organized as follows. In section 2 we present the necessary and sufficient conditions for the controllability, observability, stability and asymptotic stability for the Autonomous Discrete Matrix Sylvester Systems after introducing the concepts. Section 3 deals with rank based conditions for the Discrete Linear Sylvester Repetitive Process. Section 4 is concerned with characterizing controllability properties to design control schemes based on linear matrix inequality (LMI) for Discrete

Linear Sylvester Repetitive Process as well as corresponding process in the presence of uncertainty in the coefficient matrices.

2 Discrete Matrix Sylvester Systems

In this section first we consider discrete matrix Sylvester system with constant coefficient matrices

$$\Delta T(n) = AT(n) + T(n)B + AT(n)B + KU(n), T(0) = T_0, \tag{2.1}$$

$$Z(n) = CT(n) \tag{2.2}$$

where A, B, C and K are constant matrices of order $s \times s$, $s \times s$, $r \times s$ and $s \times m$ and the control function $U(n)$ is a matrix of order $m \times s$ respectively whose elements are real or complex functions defined on $N_{n_0}^+$ (or C), $T(n) \in \{N_{n_0}^+\}^{s\times s}$ $(or\ C^{s\times s})$ and the output $Z(n) \in \{N_{n_0}^+\}^{r\times s}$ $(or\ C^{r\times s})$. We define controllablity of discrete matrix Sylvester system as

Definition 1. The discrete matrix Sylvester system (2.1)-(2.2) is said to be completely state controllable if for any initial time n_0 and initial state $T(n_0)$, there exists a set of unconstrained controls U(0), U(1), ..., U(N-1) such that the system (1.1)-(1.2) is driven from T(n_0)=T_0 to $T(N) = T_f$ and any given state T_f there exists a finite time N > n_0. More over the set of all controllable states on [0,N] is said to be a controllable set on [0,N], denoted by ℓ[0,N].

For proofs of the theorems1 to 4 stated in this section refer [34]. The following theorem gives necessary and sufficient conditions for controllability discrete matrix Sylvester system:

Theorem 1. The discrete matrix Sylvester System (1.1)-(1.2) is completely state controllable if and only if the s×Nm matrix $R_c = [K,(A+I)K,(A+I)^2K,...,(A+I)^{N-1}K]$ and the the Nm×s matrix $S_c = [K,(B^*+I)K,((B^*+I)^2K,...,(B^*+I)^{N-1}K]^*$ have full rank.

Define observability of discrete matrix Sylvester system as

Definition 2. The autonomous discrete matrix Sylvester system (1.1)-(1.2) is said to be completely observable on $[n_0,N]$ if any initial state $T(n_0) = T_0$ is uniquely determined by the corresponding output $Z(n)$ for $n \in [n_0,N]$.

Necessary and sufficient conditions for controllability discrete matrix Sylvester system are presented in the following theorem:

Theorem 2. The autonomous discrete matrix Sylvester system (1.1)-(1.2) is completely state observable if and only if the Nr×s matrix $R_o = [C^*,(A^*+I)C^*,(A^*+I)^2C^*,...,(A^*+I)^{N-1}C^*]^*$ and the the $s \times Nr$ matrix $S_o = [C^*,(B+I)C^*,(B+I)^2C^*,...,(B+I)^{N-1}C^*]$ have full rank.

Consider the homogeneous discrete matrix Sylvester system

$$\Delta T(n) = AT(n) + T(n)B + AT(n)B \tag{2.3}$$

satisfying the initial condition $T(n_0) = T_0$. Define stability and asymptotic asymptotically of discrete matrix Sylvester system as

Definition 3. The solution $T(n) = 0$ of the discrete matrix Sylvester system (2.1) is said to be stable if given an $\varepsilon > 0$, there is a $\delta(\varepsilon, n_0)$ such that for any $T_0 \in B_\delta$ the solution $T_n \in B_\varepsilon$.

Definition 4. The solution $T(n) = 0$ of the discrete matrix Sylvester system (2.1) is said to be asymptotically stable if it is stable and $\lim_{n\to\infty} T(n) = 0$.

The following theorems present necessary and sufficient conditions for stability and asymptotic asymptotically of discrete matrix Sylvester system.

Theorem 3. The solution $T = 0$ of the homogeneous discrete matrix Sylvester system (2.3) is asymptotically stable if and only if the eigenvalues of $(A+I)$ and $(B+I)$ are inside the unit disk.

Theorem 4. The solution $T = 0$ of the homogeneous autonomous discrete matrix Sylvester system (2.3) is stable if and only if the eigenvalues of $(A+I)$ and $(B+I)$ have modulus less than 1 and those of modulus 1 are semisimple.

We present basics of vectorization and kronecker product of matrices which will be used to prove some important theorems in the following sections. Vectorization of a matrix is defined as

Definition 5. Let $A = [a_{ij}] \in C^{r\times s}(R^{r\times s})$, we denote $\widehat{A}$=Vec A = $[A_{.1}, A_{.2}, ..., A_{.s}]^T$, where $A_{.j}$=$[a_{1j}, a_{2j}, ..., a_{rj}]^T$, $(1 \le j \le s)$.

The kronecker product of matrices has the following properties.

1. $(A\otimes B)^* = A^* \otimes B^*$
2. $(A\otimes B)^{-1} = A^{-1} \otimes B^{-1}$
3. $(A\otimes B)(C\otimes D) = (AC\otimes BD)$ This rule holds, provided the dimensions of the matrices are such that expressions are defined
4. $\|(A\otimes B)\| = \|A\|\|B\|$ (where $\|A\| = \underset{i,j}{Max}|a_{ij}|$)
5. $(A+B)\otimes C = (A\otimes C)+(B\otimes C)$.
6. If $A(n)$ and $B(n)$are matrices and Δ is the forward difference operator then
   $\Delta(A\otimes B) = (\Delta A)\otimes B + A\otimes(\Delta B) + (\Delta A)\otimes(\Delta B)$
7. $Vec(ATB) = (B^*\otimes A)VecT$
8. If A and B are square matrices of order "s", then
   $(i)Vec(AT) = ((I_s\otimes A)Vec(T)$
   $(ii)Vec(TB) = ((B^*\otimes I_s)Vec(T)$.

3 Rank Based Conditions

This section is concerned with the rank based conditions for stability and asymptotic stability of Discrete linear repetitive processes (1.1)- (1.2).

Definition 6. Discrete linear repetitive processes (1.1)- (1.2) are said to be pass profile controllable if there exists a pass number q^* and control input matrix $U_{l+1}(n)$, defined over $0 \le n \le \alpha-1, 0 \le l \le q^*$, which will drive the process to an arbitrarily specified pass profile on pass q^*.

Definition 7. Discrete linear repetitive processes (1.1)- (1.2) are said to be pass boundary observable in $OS = \{0, 1, ..., N\}$, if for all $n_j \in OS$ and for all controls U_l, $0 \le n \le n_j$ and boundary data $W_{l+1}, F(n)$ at initial state, for any two trajectories $T_l, \widetilde{T}_l$, $00 \le n \le n_j$ belonging to the same input U_l, $0 < n_j$, from $CT_{l+1}(n) + DU_{l+1}(n) + D_0Z_l(n)$=$C\widehat{T}_{l+1}(n) + DU_{l+1}(n) + D_0Z_l(n)$.

Now by applying vector operation to the Discrete Linear Sylvester Repetitive Process (1.1)- (1.2) we get

$$\Delta\widehat{T}_{l+1}(n) =[(I_s\otimes A)+(B^*\otimes I_s)+(B^*\otimes A)]\widehat{T}_{l+1}(n)+(I_s\otimes K)\widehat{U}_{l+1}(n)+(I_s\otimes K_0)\widehat{Z}_l(n) \tag{3.1}$$

$$\widehat{Z}_{l+1}(n) =(I_s\otimes C)\widehat{T}_{l+1}(n)+(I_s\otimes D)\widehat{U}_{l+1}(n)+(I_s\otimes D_0)\widehat{Z}_l(n), 0\leq n\leq\infty, l\geq 0 \tag{3.2}$$

Theorem 5. The discrete linear repetitive process described by (1.1)- (1.2) is Asymptotically Stable if, and only if, $\rho(D_0)<1$ where $\rho(D_0)<1$ denotes the spectral radius of the operator D_0.

Theorem 6. Suppose that the pair $\{A,B,K_0\}$ is controllable and the pair $\{C,A,B\}$ is observable. Then Discrete Linear Sylvester Repetitive Process is stable along the puss if and only if
(i) $\rho(D_0)<1$ (ii) $\rho(A)<1$, $\rho(B)<1$
where $\rho(.)$ denotes the spectral radius, and
(iii) all eigenvalues of the transfer function matrix $G(f)$ have modulus strictly less than unity for all $|f|=1$, where $G(f)=C[fI_{s^2}-[(I_s\otimes A)+(B^*\otimes I_s)+(B^*\otimes A)]^{-1}K_0+D_0$

In the literature equivalent sets of necessary and sufficient conditions for stability along the pass in the case of repetitive processes described by vector systems [28] are available but the advantage of set of conditions of theorem 6 is that they can be tested by direct application of standard (or $1D$) linear time invariant systems stability tests similar to that of theorems 2 and 4. condition (iii) of theorem 6 can be visualized as a Nyquist like graphical interpretation. Also all three of these conditions have well defined physical meanings. In the case of condition (i) of theorem 6, set $n=0$ and consider the case of $\Delta\widehat{T}_{l+1}(0)=0, l\geq 0$. Then from equation (3.2) it follows that $\widehat{Z}_{l+1}(0)=(I_s\otimes D_0)^n, l\geq 0$. To satisfy this condition, as the process evolves from pass to pass (i.e. in the n direction) the sequence of pass initial conditions does not become unbounded in a well defined sense. Also condition (ii) of theorem 6 is a necessary condition to guarantee that the dynamics produced along any pass are uniformly bounded independent of the pass length.

Consider now the case of equations (3.1) and (3.2) with zero state initial conditions and control inputs on each pass. Then, from [30] it can be observed that the dynamics of this Discrete Linear Sylvester Repetitive Process can be written in standard (or $1D$) z transform terms as $\widehat{Z}_{l+1}(z)=G(z)\widehat{Z}_l(z), l\geq 0$. Consider single-input/single-output (SISO) and $|z|=1$, it follows that $\widehat{Z}_{l+1}(z)=G(z)\widehat{Z}_l(z), l\geq 0$, and hence condition (iii) of theorem 6 is equivalent to the requirement that each frequency component of the initial profile is attenuated from pass to pass. condition (i) of theorem 6 is a necessary and sufficient condition for a asymptotic stability of Discrete Linear Sylvester Repetitive Process. If the input sequence applied $\{\widehat{U}_l(n)\}_{l\geq 1}$, converges strongly (i.e. in the norm topology of the underlying Banach space) then under asymptotic stability, the sequence of pass profiles $\{\widehat{Z}_l\}_{l\geq 1}$, generated converges strongly to the following limit profile described by the standard (i.e. $1D$) state space model similar to the one in described in [15]

$$\begin{aligned}\Delta\widehat{T}_\alpha(n) =&[\{[(I_s\otimes A)+(B^*\otimes I_s)+(B^*\otimes A)]+(I_s\otimes K_0)[I_{ms}-(I_s\otimes D_0)]^{-1}(I_s\otimes C)\}]\widehat{T}_\alpha(n)+\\ &[(I_s\otimes K)+(I_s\otimes K_0)[I_{ms}-(I_s\otimes D_0)]^{-1}(I_s\otimes D)]\widehat{U}_\alpha(n)\end{aligned} \tag{3.3}$$

$$\widehat{Z}_\alpha(n) =[I_{ms}-(I_s\otimes D_0)]^{-1}(I_s\otimes C)\widehat{T}_\alpha(n)+[I_{ms}-(I_s\otimes D_0)]^{-1}(I_s\otimes D)\widehat{U}_\alpha(n), 0\leq n\leq\alpha \tag{3.4}$$

In physical terms this means that if a Discrete Linear Sylvester Repetitive Process is asymptotically stable then, after a sufficiently large number of passes, its repetitive dynamics can be replaced by those of a standard, or $1D$ discrete linear system. This fact has obvious implications in terms of controlling the dynamics of Discrete Linear Sylvester Repetitive Process. Note, however, that asymptotic stability does not guarantee that the resulting limit profile is stable in the $1D$ linear sense which is clearly what would be required in most applications. Stability along the pass is required to guarantee that the limit profile resulting from equations (3.1) and (3.2) is stable in the $1D$ sense. In particular, from [28] it follows that the conditions of Theorem 6 imply that

$$\rho\{[(I_s \otimes A) + (B^* \otimes I_s) + (B^* \otimes A)] + (I_s \otimes K_0)[I_{ms} - (I_s \otimes D_0)]^{-1}(I_s \otimes C)\} < 1.$$

The conditions of theorem 6 can be tested by direct application of an suitable set of $1D$ linear systems stability tests. From the works of K.J. Smyth [30] and E. Rogers, D.H. Owens [28] it follows that, if the process is stable along the pass it is possible to derive computable performance bounds on the following key measures of system performance. The rate of approach of the output sequence of pass profiles $\{\widehat{Z}_l\}_{l\geq 1}$ to the limit profile $\widehat{Z}_\alpha$ and the error $(\widehat{Z}_l - \widehat{Z}_\alpha)$ on pass $k \geq 0$. By expressing these bounds in terms of the norm on the underlying Banach space these bounds are computed from the $1D$ linear system parameterized by the state space quadruple $\{[(I_s \otimes A) + (B^* \otimes I_s) + (B^* \otimes A)], (I_s \otimes K_0), (I_s \otimes C), (I_s \otimes D_0)\}$ which describes the contribution of the dynamics of the previous pass profile to that of the current one. The computational algorithms follow from combining basic results in functional analysis with the theory of nonnegative matrices.

Attempts to control these processes using classical approaches like systems theory, algorithms fail in majority of cases except in a few very restrictive special cases, because all these approaches ignores their inherent two-dimensional (2-D) systems structure, i.e., information propagation occurs from pass-to-pass and along a given pass, and also the effects of resetting the pass initial conditions before the start of each new pass.

A rigorous stability theory based Banach space setting for linear repetitive processes has been developed by many authors. One unique feature of repetitive processes is that it is possible define physically meaningful control laws for them. For example, in the iterative learning control application, one such family of control laws is composed of, state or output based, feedback control action on the current pass combined with information fed forward from the previous pass, or trial in the iterative learning control context has already been generated and is therefore available for use. It is always desired to design control laws for stability so that performance can be guaranteed.

4 LMI Based

Analysis using Discrete Linear Sylvester Repetitive Processes have distinct advantages over alternatives. Examples of these algorithmic applications include classes of iterative learning control (ILC) schemes [23] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [24]. The stability theory for discrete linear repetitive processes is the essential basis for a rigorous stability/ convergence theory for a class of algorithms in ILC schemes.

In the general case of repetitive processes it is highly desirable to have an analysis setting where such control laws can be designed for either stability or performance or both. From the literature [33], [23], [16] it is observed that an LMI re-formulation of the stability conditions for discrete linear repetitive processes leads naturally to design algorithms to ensure closed loop stability along the pass under a control law which explicitly makes use of the current pass state vector. Implementation of such a control law require an observer to provide the current pass state vector component. LMI setting is used to design control laws which only require pass profile information (which has already been generated and hence is available control action) for implementation as an alternative. LMI based methods have also been investigated as a means of stability analysis and controller design for 2D discrete linear systems described [27] and [12] state space models. To complete the process description, it is necessary to specify the boundary conditions, i.e. the state initial matrix on each pass and the initial pass profile. Here no loss of generality arises from assuming

$$T_{l+1}(0) = W_{l+1}, l \geq 0 \tag{4.1}$$

$$Z_0(n) = F(n) \tag{4.2}$$

where W_{l+1} is the $s \times s$ matrix W_{l+1} has known constant entries and $F(n)$ is an $m \times s$ matrix whose entries are known functions of n over $[0, \infty]$. The stability of Discrete Linear Sylvester Repetitive Process along the pass is relevent as well as important, necessary and sufficient conditions for vector Repetitive Process to have this property are known [28], but here we present the sufficient condition for stability of Discrete Linear Sylvester Repetitive Process along the pass which allows us to design control laws in a straightforward manner.

Throughout this section we consider vectorized form of Discrete Linear Sylvester Repetitive Process (1.1)- (1.2) given by (3.1-(3.2) for proving various results. The boundary conditions specified by (4.1) and (4.2) can be vectorized by using the definition of kronecker product defined in Section 2 and then plugged in the proofs where they are required in this section.

Theorem 7. Discrete Linear Sylvester Repetitive Process processes are stable along the pass if there exist matrices $W = W_1 \oplus W_2 > 0$ and $Q > 0$ which solve the following equation

$$\widehat{A}^T W^{1,0} + W^{1,0}\widehat{A} + \widehat{A}^T W^{1,0}\widehat{A} - W^{0,1} = -Q < 0 \tag{4.3}$$

where $\oplus$ denotes the direct sum, i.e., $W = diag\{W1, W2\}$, 0_p the null matrix of dimension $p \times p$, and

$$\widehat{A} = \begin{bmatrix} \overline{A} & B_0 \\ C & D_0 \end{bmatrix}, W^{1,0} = W_1 \oplus O_{ms}, W^{0,1} = O_{s^2} \oplus W_2 \tag{4.4}$$

$$\overline{A} = [(I_s \otimes A) + (B^* \otimes I_s) + (B^* \otimes A)]. \tag{4.5}$$

Proof. Proof follows by discretizing the results of E. Rogers and D. H. Owens [28].

In the single-input single-output (SISO) case, it can be observed that the condition (4.3) of the Theorem 7 is both necessary and sufficient for stability along the pass.

The equation (4.3) in the Theorem 7 can be written as

$$\widehat{A_2}^T \widehat{W_2}\widehat{A_2} - W^{0,1} + \widehat{A_1}^T W^{1,0} + W^{1,0}\widehat{A_1} < 0 \tag{4.6}$$

where $\widehat{W_2} = W_3 \oplus W_2$, the $s^2 \times s^2$ matrix $W_3 > 0$ is arbitrary and

$$\widehat{A_1} = \begin{bmatrix} \overline{A} & B_0 \\ 0 & 0 \end{bmatrix}, \widehat{A_2} = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}. \tag{4.7}$$

Now the following equivalent condition is obtained by applying the Schur complements formula to the equation (4.6) with $W = W^{0,1} + \widehat{A_1}^T W^{1,0} + W^{1,0}\widehat{A_1}$, $L = \widehat{A_2}$, and $V = \widehat{W_2}$, and then by pre and post multiply the result matrix by the matrix $[I + \widehat{W_2}]$

$$\begin{bmatrix} -W^{0,1} + \widehat{A_1}^T W^{1,0} + W^{1,0}\widehat{A_1} & \widehat{A_2}^T \widehat{W_2} \\ \widehat{W_2}\widehat{A_2} & -\widehat{W_2} \end{bmatrix} < 0, \tag{4.8}$$

This condition is in the form of LMI and hence theorem 8 follows.

Theorem 8. Discrete Linear Sylvester Repetitive Process (1.1)-(1.2) is stable along the pass if the linear matrix inequality (LMI) given by (4.8) is feasible.

To design a controller for Discrete Linear Sylvester Repetitive Process, we consider a control law of the form

$$\Delta U_l(n) = P_1\Delta T_l(n) + P_2 Z_l(n) = P\begin{bmatrix}\Delta T_l(n)\\ Z_l(n)\end{bmatrix} \quad \text{for} \quad 0 \le n \le \infty,\ l \ge 0 \tag{4.9}$$

This control law uses feedback of the current pass state vector, which is assumed to be available for use here, and feedforward of the previous pass-profile vector. By using the control law (4.9) in the similar lines that of [28], we obtain

$$\mathcal{C}(f,z) \neq 0, \quad \text{for all } (f,z),\, Re(f) \ge 0,\, |z| \le 1 \tag{4.10}$$

where

$$\mathcal{C}(f,z) = fI_{s^2} \oplus I_{ms} - \begin{bmatrix} (\overline{A} + (I_s\otimes K)(I_s\otimes P_1)) & (I_s\otimes K_0) + (I_s\otimes K)(I_s\otimes P_2) \\ f(I_s\otimes C) + (I_s\otimes D)(I_s\otimes P_1) & z(I_s\otimes C) + (I_s\otimes D)(I_s\otimes P_1)\end{bmatrix}. \tag{4.11}$$

Now Introduce the matrices

$$\widehat{K}_1 = \begin{bmatrix} I_s\otimes K \\ 0\end{bmatrix},\ \widehat{K}_2 = \begin{bmatrix} 0 \\ I_s\otimes D\end{bmatrix}. \tag{4.12}$$

Now by using linear matrix inequality (LMI) we obtain conditions for closed loop stability along the pass coupled with applied algorithms for computing the controller parameters.

Theorem 9. Consider a Discrete Linear Sylvester Repetitive Process (1.1)-(1.2) subject to a control law defined by (4.9), then the resulting closed-loop process is stable along the pass if there exist matrices $Y_1 > 0$, $Y_2 > 0$ together with matrices N_1 and N_2, such that the following LMI holds:

$$\begin{bmatrix} Column1 & Column2 & Column3 \end{bmatrix} < 0 \tag{4.13}$$

where

$$Column1 = \begin{bmatrix} (Y_1\otimes I_s)\overline{A}^T + (N_1^T\otimes I_s)(I_s\otimes K^T) + \overline{A}(Y_1\otimes I_s) + (I_s\otimes K)(I_s\otimes N_1) \\ (I_s\otimes Y_2) + (I_s\otimes K_0^T) + (I_s\otimes N_2^T)(I_s\otimes K^T) \\ (I_s\otimes C)(I_s\otimes Y_1) + (I_s\otimes D)(I_s\otimes N_1) \end{bmatrix}$$

$$Column2 = \begin{bmatrix} (I_s\otimes K_0)(I_s\otimes Y_2) + (I_s\otimes K)(I_s\otimes N_2) \\ -(I_s\otimes Y_2) \\ (I_s\otimes D_0)(I_s\otimes Y_2) + (I_s\otimes D)(I_s\otimes N_2) \end{bmatrix}$$

$$Column3 = \begin{bmatrix} (I_s\otimes Y_1)(I_s\otimes C^T) + (I_s\otimes N_1^T)(I_s\otimes D^T) \\ (I_s\otimes Y_2)(I_s\otimes D_0^T) + (I_s\otimes N_2^T)(I_s\otimes D^T) \\ -(I_s\otimes Y_2) \end{bmatrix}.$$

If the above LMI defined by (4.13) holds, stabilizing P_1 and P_2 for the control law defined by equation (4.9) are given by

$$P_1 = (I_s\otimes N_1)(I_s\otimes Y_1)^{-1} \text{ and } P_2 = (I_s\otimes N_2)(I_s\otimes Y_2)^{-1}. \tag{4.14}$$

Proof. From the LMI condition (4.8) and equation (4.9), the closed-loop Discrete Linear Sylvester Repetitive Process (1.1)-(1.2) is stable along the pass if there exist symmetric matrices $W_1 > 0$ and $W_2 > 0$, such that the following equation is satisfied

$$\begin{bmatrix} -W^{0,1} + (\widehat{A}_1 + \widehat{K}_1 P)^T W^{1,0} + W^{1,0}(\widehat{A}_1 + \widehat{K}_1 P) & (\widehat{A}_2 + \widehat{K}_2 P)^T \widehat{W}_2 \\ \widehat{W}_2(\widehat{A}_2 + \widehat{K}_2 P) & -\widehat{W}_2 \end{bmatrix} < 0. \tag{4.15}$$

The above matrix inequality (4.15) is nonlinear in its parameters and by substituting relevant sub matrices in (4.15) a strict LMI form of inequality is obtained

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix} < 0 \tag{4.16}$$

where

$$e_{11} = [\overline{A} + (I_s \otimes K)(I_s \otimes P_1)]^T W_1 + W_1[\overline{A} + (I_s \otimes K)(I_s \otimes P_1)], \tag{4.17}$$

$$e_{12} = W_1[(I_s \otimes K_0) + (I_s \otimes K)(I_s \otimes P_2)], e_{13} = 0, e_{14} = [(I_s \otimes C) + (I_s \otimes D)(I_s \otimes P_1)]^T, \tag{4.18}$$

$$e_{21} = [(I_s \otimes K_0) + (I_s \otimes K)(I_s \otimes P_2)]^T W_1, e_{22} = -W_2, e_{23} = 0, \tag{4.19}$$

$$e_{24} = [(I_s \otimes D_0) + (I_s \otimes D)(I_s \otimes P_2)]^T W_2, e_{31} = 0, e_{32} = 0, e_{33} = -W_2, e_{34} = 0, \tag{4.20}$$

$$e_{41} = W_2[(I_s \otimes C) + (I_s \otimes D)(I_s \otimes P_1)], e_{42} = W_2[(I_s \otimes D_0) + (I_s \otimes D)(I_s \otimes P_2)], e_{43} = 0, e_{44} = -W_2 \tag{4.21}$$

By pre and post multiplying the equation (4.16) by the matrix $\begin{bmatrix} W_1^{-1} & 0 & 0 & 0 \\ 0 & W_2^{-1} & 0 & \\ 0 & 0 & W_2^{-1} & 0 \\ 0 & 0 & 0 & W_2^{-1} \end{bmatrix}$, it is observed that third block row and the third block column of the resulting matrix are removed without changing the underlying inequality solution. Now Substitution of $(I_s \otimes Y_1) = W_1^{-1}$ and $(I_s \otimes Y_2) = W_2^{-1}$ and realization of equation (4.14) yields equation (4.13)

We consider Discrete Linear Sylvester Repetitive Process in the presence of uncertainty in the model structure. This can be done by imposing uncertainty structures on the coefficient matrices of the Repetitive Process. We extend the LMI-based stability and controller design analysis to two cases where the uncertainty is so expressed.

From the literature [19], it is observed that commonly used uncertainty descriptions for robust control studies are either norm bounded or of a polytopic form in both $1D$ continuous and discrete linear systems. These two representations have theoretical and application importance along with their advantages and disadvantages. Discrete linear repetitive processes have strong structural links with $1D$ discrete linear systems. For the considered Discrete Linear Sylvester Repetitive Process the links with $1D$ discrete linear systems are associated with the structure of the linear difference equation governing the along the pass state dynamics from the updating of the pass-profile matrix from pass-to-pass. Based on this, we consider (i) the uncertainty in the current pass state dynamics updating has a polytopic form and the pass-to-pass updating uncertainty is norm bounded (ii) both uncertainty structures are assumed to be norm bounded.

Case(i)

In this case, we assume that the uncertainty in the Discrete Linear Sylvester system which governs the evolution of the current pass state matrix equation in (1.1)-(1.2) has a polytopic character. It is assumed that all possible choices for the matrices which define Discrete Linear Sylvester system can be expressed as

$$[\overline{A}\ (I_s \otimes K)\ I_s \otimes K_0)] \in \mathbb{C}_o[\overline{A}^i\ (I_s \otimes K)^i\ I_s \otimes K_0)^i],\ i = 1,2,...,h \text{ where} \tag{4.22}$$

$$\mathbb{C}_o[\overline{A}^i\ (I_s \otimes K)^i\ I_s \otimes K_0)^i] = \left\{\mathbb{T}/\mathbb{T} = \textstyle\sum_{i=1}^h \alpha_i[\overline{A}^i\ (I_s \otimes K)^i\ I_s \otimes K_0)^i],\ \alpha_i \geq 0, \sum_{i=1}^h \alpha_i = 1\right\} \tag{4.23}$$

where $\overline{A}^i\ (I_s \otimes K)^i\ I_s \otimes K_0)^i$ are known constant matrices of suitable orders. Consider perturbed current pass-profile updating equation of (1.2) where perturbations are bounded

$$Z_{l+1}(n) = (C + \delta C)T_{l+1}(n) + (D + \delta D)U_{l+1}(n) + (D_0 + \delta D_0)Z_l(n) \text{ where} \tag{4.24}$$

$$[\delta C\ \delta D\ \delta D_0] = HF[E_1^1\ E_1^2\ E_2] \text{ and} \tag{4.25}$$

$$F^T F \leq I_{s^2} \tag{4.26}$$

where H, E_1^1, E_1^2, E_2 are known matrices of compatible orders. From the information about process the constant elements of the matrix F are to be determined.

Case(ii)

In this case we consider a norm bounded perturbation in current pass state and pass profile update equations. Now the perturbed Discrete Linear Sylvester Repetitive system is given by

$$\begin{aligned}\Delta T_{l+1}(n) =& (A + \delta A)T_{l+1}(n) + T_{l+1}(n)(B + \delta B) + A + \delta A)T_{l+1}(n)(B + \delta B) + \\ & (K + \delta K)U_{l+1}(n) + (K_0 + \delta K_0)Z_l(n)\end{aligned} \tag{4.27}$$

along with the equation (4.24), where

$$\begin{bmatrix}\delta A & \delta K_0 & \delta K \\ \delta C & \delta D_0 & \delta D\end{bmatrix} = \begin{bmatrix}H_1 \\ H_2\end{bmatrix} F[E_1^1\ E_1^2\ E_2] \tag{4.28}$$

and equation (4.26) is assumed to be satisfied.

Theorem 10. The perturbed current pass-profile updating equation of (1.2) where perturbations are bounded given by (4.24) the form of Case (i) is stable along the pass if there exist matrices $W_1 > 0$ and $W_2 > 0$, which satisfy the following set of LMIs

$$[\widehat{A}_1^i]^T W^{1,0} + W^{1,0}[\widehat{A}_1^i] - W^{0,1} + [\widehat{A}_2 + \widetilde{H}\widetilde{F}\widetilde{E}_1]^T \widehat{W}_2[\widehat{A}_2 + \widetilde{H}\widetilde{F}\widetilde{E}_1] < 0,\ i = 1,2,...,h \tag{4.29}$$

where $\widehat{A}_2$ is given by (4.7) and

$$\widehat{A}_1^i = \begin{bmatrix}\overline{A}^i & (I_s \otimes K_0)^i \\ 0 & 0\end{bmatrix}, \widetilde{H} = \begin{bmatrix}0 & 0 \\ H & H\end{bmatrix}, \widetilde{F} = \begin{bmatrix}F & 0 \\ 0 & F\end{bmatrix}, \widetilde{E}_1 = \begin{bmatrix}E_1^1 & 0 \\ 0 & E_1^2\end{bmatrix} \tag{4.30}$$

Proof. If stability property holds for each vertex model $\overline{A}^i$, K_0^i, K^iC, D_0 and D then stability for the considered polytopic uncertainty can be achieved. Proof follows from achieving stability for each vertex model in usual way.

Though this results is theoretically interesting but difficult to apply because of the presence of uncertainty matrix F which is not in exact LMI form. To overcome this difficulty we present the following result for Discrete Linear Sylvester Repetitive Process in the similar lines that of P. P. Khargonekar et al [19].

Lemma 1. The LMI (4.29) hold if, and only if, there exists a scalar $\varepsilon > 0$ and matrices $W_1 > 0$ and $W_2 > 0$, such that for $i = 1,2,...,h$,

$$\begin{bmatrix} -\widehat{W}_2^{-1} + \varepsilon \widetilde{H}\widetilde{H}^T & \widehat{A}_2 \\ \widehat{A}_2^T & \varepsilon^{-1}\widehat{E}_1^T\widehat{E}_1 + [\widehat{A}_1^i]^T W^{1,0} + W^{1,0}[\widehat{A}_1^i] - W^{0,1} \end{bmatrix} < 0 \tag{4.31}$$

Theorem 11. The perturbed current pass-profile of Discrete Linear Sylvester Repetitive Process equation (4.24) stated in Case(i) is stable along the pass if there exist matrices $W_1 > 0$ and $W_2 > 0$ and a real scalar $\mu > 0$, such that the following set of LMIs is feasible for $i = 1,2,...,h$

$$\begin{bmatrix} -\widehat{W}_2 & \widehat{W}_2\widehat{A}_2 & \widehat{W}_2\widetilde{H} & 0 \\ \widehat{A}_2^T\widehat{W}_2 & [\widehat{A}_1^i]^T W^{1,0} + W^{1,0}[\widehat{A}_1^i] - W^{0,1} & 0 & \mu\widehat{E}_1^T \\ \widetilde{H}^T\widehat{W}_2 & 0 & -\mu I & 0 \\ 0 & \mu\widehat{E}_1 & 0 & -\mu I \end{bmatrix} < 0 \tag{4.32}$$

Proof. By applying Schurs complement formula to inequality (4.31) with

$$W = \begin{bmatrix} -\widehat{W}_2 & \widehat{W}_2\widehat{A}_2 \\ \widehat{A}_2^T\widehat{W}_2 & [\widehat{A}_1^i]^T W^{1,0} + W^{1,0}[\widehat{A}_1^i] - W^{0,1} \end{bmatrix}, V = \begin{bmatrix} \varepsilon I & 0 \\ 0 & \varepsilon^{-1} I \end{bmatrix}, L = \begin{bmatrix} \widetilde{H}^T & 0 \\ 0 & \widehat{E}_1 \end{bmatrix} \tag{4.33}$$

then by pre and post multiplying the resultant inequality by the matrix $\begin{bmatrix} \widehat{W}_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \varepsilon^{-1} I \end{bmatrix}$ and taking $\varepsilon^{-1} = \mu$ inequality (4.32) can be realized.

Introduce the following notation for the consideration of Case(ii)

$$\delta\widehat{A}_1 = \begin{bmatrix} \delta\overline{A} & \delta K_0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} H_1 \\ 0 \end{bmatrix} F [E_1^1 \ E_1^2] = \widetilde{H}_1 F \widehat{E} \tag{4.34}$$

and

$$\delta\widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ \delta C & \delta D_0 \end{bmatrix} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} F [E_1^1 \ E_1^2] = \widetilde{H}_2 F \widehat{E} \tag{4.35}$$

We also sate a lemma [19] important for the future discussion

Lemma 2. Let Σ_1, Σ_2 and Ω be real matrices of appropriate dimensions. Then for any $\Omega^T\Omega \leq I$ and scalar $\varepsilon > 0$ the following inequality is satisfied:

$$\Sigma_1\Omega\Sigma_2 + \Sigma_2^T\Omega\Sigma_1^T \leq \varepsilon^{-1}\Sigma_1\Sigma_1^T + \varepsilon\Sigma_2^T\Sigma_2 \tag{4.36}$$

Theorem 12. The perturbed current pass-profile of Discrete Linear Sylvester Repetitive Process (4.24) and (4.27), stated in Case(ii) is stable along the pass if there exist constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, matrices $W_1 > 0$, $W_2 > 0$ and $W_3 > 0$ such that the following LMI is feasible

$$\begin{bmatrix} -W^{0,1} + [\widehat{A_1^i}]W^{1,0} + W^{1,0}\widehat{A} - 1 + \varepsilon_1 \widehat{E}^T \widehat{E} & \widehat{A}_2^T \widehat{W}_2 & W^{1,0}\widetilde{H}_1 & \widehat{W}_2\widehat{H}_2 \\ \widehat{W}_2\widehat{A}_2 & -\widehat{W}_2 + \varepsilon_2 \widehat{E}^T \widehat{E} & 0 & 0 \\ \widetilde{H}_1^T W^{1,0} & 0 & -\varepsilon_1 I & 0 \\ \widehat{H}_2^T & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{4.37}$$

Proof. From Theorem 8 and linear matrix inequality (LMI) given by (4.8) it follows that stability along the pass is achieved if

$$\begin{bmatrix} -W^{0,1} + (\widehat{A_1} + \widehat{\delta A_1})^T W^{1,0} + W^{1,0}(\widehat{A_1} + \widehat{\delta A_1}) & (\widehat{A_2} + \widehat{\delta A_2})^T \widehat{W_2} \\ \widehat{W_2}(\widehat{A_2} + \widehat{\delta A_2}) & -\widehat{W_2} \end{bmatrix} < 0, \tag{4.38}$$

After decomposing the matrix in the above LMI (4.38) we get,

$$\begin{bmatrix} -W^{0,1} + \widehat{A_1}^T W^{1,0} + W^{1,0}\widehat{A_1} & \widehat{A_2}^T \widehat{W_2} \\ \widehat{W_2 A_2} & -\widehat{W_2} \end{bmatrix} + \begin{bmatrix} \widehat{\delta A_1}^T W^{1,0} + W^{1,0}\widehat{\delta A_1} & \widehat{\delta A_2}^T \widehat{W_2} \\ \widehat{W_2 \delta A_2} & 0 \end{bmatrix} < 0, \tag{4.39}$$

Now LMI (4.37) of lemma (2) gives

$$\begin{bmatrix} \widehat{\delta A_1}^T W^{1,0} + W^{1,0}\widehat{\delta A_1} & \widehat{\delta A_2}^T \widehat{W_2} \\ \widehat{W_2 \delta A_2} & 0 \end{bmatrix} \leq \begin{bmatrix} [\varepsilon_1^{-1} W^{1,0}\widehat{H}_1 \widehat{H}_1^T W^{1,0} + \\ \varepsilon_2^{-1}\widehat{W}_2 \widehat{H}_2 \widehat{H}_2^T \widehat{W}_2 + \varepsilon_1 E^T E] & 0 \\ 0 & \varepsilon_2 E^T E \end{bmatrix} \tag{4.40}$$

Now proof follows from Schur complement and congruence transform.

The results pertaining to stability established in this section can also be applied to the resulting state space model to give a sufficient condition for closed-loop stability along the pass and corresponding control law can be designed for perturbed Discrete Linear Sylvester Repetitive Process considered in both cases. The results presented in this sections are for uncertainty matrices whose elements are constants but can be extend to discrete time varying matrices.

5 Conclusions

Linear discrete Sylvester repetitive processes are a distinct class of $2D$ discrete linear systems which are of both applications and systems theoretic interest. This paper established necessary and sufficient conditions for rank based necessary and sufficient conditions for stability and asymptotic stability of Discrete linear repetitive processes.

The conditions are established by using the concepts of vectorization and are highly desirable from an applications standpoint. Linear discrete Sylvester repetitive processes, which have well defined physical interpretation for applications in ILC, cannot be analyzed/controlled by direct application of existing methods and design of control schemes for these systems is presently not available in the literature.

LMI formulation of stability along the pass, which are having importance in designing class of control laws are constructed. The developed theory and established results represent the systematic procedure for stability analysis and controller design, as opposed to just stability analysis.

Finally, we demonstrated that it is possible to control these processes in the presence of uncertainty in the model structure and provide a bench mark for future work in Discrete Linear Sylvester Repetitive Process.

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